Convex approximations for totally unimodular integer recourse models: A uniform error bound

Ward Romeijnders · Maarten H. van der Vlerk · Willem K. Klein Haneveld

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Abstract We consider a class of convex approximations for totally unimodular (TU) integer recourse models and derive a uniform error bound by exploiting properties of the total variation of the probability density functions involved. For simple integer recourse models this error bound is tight and improves the existing one by a factor 2, whereas for TU integer recourse models this is the first nontrivial error bound available. The bound ensures that the performance of the approximations is good as long as the total variations of the densities of all random variables in the model are small enough.

Keywords Stochastic Programming · Integer recourse · Convex approximation

Mathematics Subject Classification (2000) 90C15 · 90C10

1 Introduction

We consider the two-stage integer recourse problem

$$\min_x \{cx + Q(z) : Ax \geq b, z = Tx,\, x \in \mathbb{R}^{n_1}_+\}, \tag{1}$$

where z are tender variables, Q is the recourse (expected value) function

$$Q(z) := \mathbb{E}_\omega \Big[v(\omega - z) \Big], \quad z \in \mathbb{R}^m,$$

and v is the second-stage value function

$$v(s) := \min_y \{qy : Wy \geq s,\, y \in \mathbb{Z}^{n_2}_+\}, \quad s \in \mathbb{R}^m.$$

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The second-stage decision variables y represent the so-called recourse actions that compensate for infeasibilities with respect to the random goal constraints $Tx \geq \omega$. Here, the right-hand side ω is a random vector with known distribution. The functions Q and v represent the (expected) recourse cost associated with the recourse actions y.

Modeling indivisibilities or on/off decisions typically requires integer (or binary) decision variables. For this reason, introducing such integer variables to the model is highly relevant for practice, but at the same time makes the model considerably more difficult to solve. Most exact solution methods combine ideas behind algorithms designed for either stochastic continuous or deterministic integer programs. Although substantial progress has been made, in general these algorithms have difficulties solving large real-life problem instances. For an overview of these algorithms we refer to the survey papers Klein Haneveld and Van der Vlerk [5], Louveaux and Schultz [6], Schultz [11], and Sen [12].

The main reason that integer recourse models are considerably more difficult to solve than continuous recourse models is that the integer recourse function Q is generally *non-convex* [8]. A possible approach to deal with this difficulty is to construct convex approximations of the recourse function Q by modifying the recourse data (MRD) [13], which comprises the parameters and structure of the model, and the distributions of the random variables involved. The rationale for doing so is that convex optimization problems are computationally much more tractable than non-convex problems, and as long we only make small changes in the recourse data we expect to obtain close approximations.

Using MRD a class of convex approximations of Q has been developed, first for the special case of simple integer recourse models (when $W = I_m$) [4], later extended to general complete integer recourse models [14], and mixed-integer recourse models with a single recourse constraint [15]. The recurring idea in these so-called α-approximations is to simultaneously relax the integrality constraints and perturb the distribution of the right-hand side random vector ω. In this way, a difficult-to-solve integer recourse problem is approximated by a continuous recourse problem for which efficient algorithms exist such as (variants of) the L-shaped algorithm [16].

Although a uniform error bound for these approximations is available for models with a simple recourse structure [4], such an error bound is lacking for integer recourse models in general. We derive a uniform error bound for integer recourse models with a totally unimodular (TU) recourse matrix W by exploiting properties of the total variation of probability density functions. This error bound is tight for simple integer recourse models and improves the existing error bound by a factor 2. Moreover, the error bound ensures that the convex approximations are good as long as the total variations of the densities of all random variables in the model are small enough.

The remainder of this paper is organized as follows. We introduce α-approximations of integer recourse models in Section 2. To set the stage for our analysis, we discuss properties of the total variation of probability density functions in Section 3, and we solve a simplified one-dimensional bounding problem in Section 4. In Sections 5 and 6 we derive a uniform error bound for α-approximations of TU integer recourse models with independent and dependent random variables, respectively.

2 Convex approximations and literature review

Throughout this paper we use the following assumptions.

(i) W is a complete recourse matrix, that is, for every $s \in \mathbb{R}^m$ there exists $y \in \mathbb{Z}_+^{n_2}$ such that $Wy \geq s$, and thus $v(s) < +\infty$,
(ii) the recourse structure is sufficiently expensive, that is, $v(s) > -\infty$ for all $s \in \mathbb{R}^m$, and
(iii) $\mathbb{E}_\omega[|\omega|]$ is finite.

As a result the recourse function Q is finite everywhere.

We consider so-called *α-approximations* of Q, which is a class of convex approximations of Q studied in Van der Vlerk [14] and related work. These α-approximations are an example of MRD as discussed earlier.

Definition 1 For every $\alpha \in \mathbb{R}^m$, the *α-approximation of* Q is given by

$$Q_\alpha(z) := \mathbb{E}_\omega \left[\min_y \{ qy : Wy \geq \lceil \omega \rceil_\alpha - z,\, y \in \mathbb{R}_+^{n_2} \} \right], \quad z \in \mathbb{R}^m,$$

where $\lceil \omega \rceil_\alpha := \lceil \omega - \alpha \rceil + \alpha$ is the round-up of ω with respect to $\alpha + \mathbb{Z}^m$.

Remark 1 Note that the definition of α-approximations is given for $\alpha \in \mathbb{R}^m$ but since $Q_\alpha \equiv Q_{\alpha'}$ if $\alpha - \alpha' \in \mathbb{Z}^m$, we could have restricted the definition to $\alpha \in [0, 1)^m$.

For every $\alpha \in \mathbb{R}^m$, the random vector $\lceil \omega \rceil_\alpha$ is discretely distributed with support in $\alpha + \mathbb{Z}^m$. Hence, the α-approximation Q_α is the recourse function of a *continuous* recourse model with *discrete* random right-hand side vector $\lceil \omega \rceil_\alpha$, and thus Q_α is a convex polyhedral function. Although Dyer and Stougie [1] show that from a theoretical complexity point of view these problems are hard to solve in general, there exist algorithms that can solve such recourse problems involving discrete distributions within reasonable time limits. This implies that if the difference between $Q(z)$ and its approximation $Q_\alpha(z)$ is small enough for all $z \in \mathbb{R}^m$, then the approximating model is not only computationally tractable, but also leads to (near-)optimal solutions. For this reason, we use the supremum norm to measure the error of the approximations:

$$\|Q - Q_\alpha\|_\infty := \sup_{z \in \mathbb{R}^m} |Q(z) - Q_\alpha(z)|, \quad \alpha \in \mathbb{R}^m.$$

The main contribution of this paper is the derivation of nontrivial upper bounds of this error for integer recourse models with TU recourse matrix W.

In the remaining part of this section we review the existing literature on such upper bounds. First of all, consider the case $W = I_m$. Then, problem (1) reduces to a one-sided simple integer recourse (SIR) problem [7]. This problem is called simple because the recourse function $Q(z)$ is separable in the components of z, so that

$$Q(z) = \mathbb{E}_\omega \Big[\min_y \{ qy : y \geq \omega - z,\, y \in \mathbb{Z}_+^{n_2} \} \Big] = \sum_{i=1}^m q_i Q_i(z_i), \quad z \in \mathbb{R}^m, \quad (2)$$

where $Q_i(z_i) := \mathbb{E}_{\omega_i}[\lceil \omega_i - z_i \rceil^+]$, and similarly

$$Q_\alpha(z) = \sum_{i=1}^m q_i \mathbb{E}_{\omega_i} \Big[(\lceil \omega_i \rceil_{\alpha_i} - z_i)^+ \Big], \quad z \in \mathbb{R}^m.$$

Here, $(x)^+ := \max\{0, x\}$ denotes the positive part of $x \in \mathbb{R}$ (also, componentwise for $x \in \mathbb{R}^m$), and we conveniently write $\lceil x \rceil^+$ to denote $\max\{0, \lceil x \rceil\}$.

The properties of the m-dimensional SIR function Q follow directly from those of the generic one-dimensional SIR function

$$\mathcal{Q}(z) := \mathbb{E}_\omega[\lceil \omega - z \rceil^+], \quad z \in \mathbb{R}.$$

If the one-dimensional random variable ω is discretely distributed, then efficient algorithms are available to construct the convex hull of $\mathcal{Q}$ [2,3]. If ω is continuously distributed with probability density function (pdf) f of bounded variation, then Klein Haneveld et al. [4] show that for every $\alpha \in \mathbb{R}$,

$$\| \mathcal{Q} - \mathcal{Q}_\alpha \|_\infty \leq \min \left\{ \frac{|\Delta| f}{4}, 1 \right\},$$

where $\mathcal{Q}_\alpha$ denotes the α-approximation of $\mathcal{Q}$ and $|\Delta| f := |\Delta| f(\mathbb{R})$ the total variation of f on $\mathbb{R}$. This result leads to the following uniform upper bound on the error in the case of simple integer recourse,

$$\sup_{z \in \mathbb{R}^m} |Q(z) - Q_\alpha(z)| \leq \sum_{i=1}^m q_i \min \left\{ \frac{|\Delta| f_i}{4}, 1 \right\}, \quad \alpha \in \mathbb{R}^m, \quad (3)$$

where f_i is the marginal pdf of ω_i.

Let us now consider the more general case, where the recourse matrix W is TU. The second-stage value function v can be rewritten in a more convenient form. Since the recourse is complete and sufficiently expensive, we have for all $s \in \mathbb{R}^m$,

$$\begin{aligned} v(s) &= \min_y \{ qy : Wy \geq s,\, y \in \mathbb{Z}_+^{n_2} \} \\ &= \min_y \{ qy : Wy \geq \lceil s \rceil,\, y \in \mathbb{R}_+^{n_2} \} && (4) \\ &= \max_\lambda \{ \lambda \lceil s \rceil : \lambda W \leq q,\, \lambda \in \mathbb{R}_+^m \}, && (5) \end{aligned}$$

where the equality in (4) follows from the fact that W is TU, and the equality in (5) holds by strong LP duality. Assumptions (i) and (ii) also imply that the dual feasible region $\{\lambda W \leq q, \lambda \geq 0\}$ is non-empty and bounded. Thus it is spanned by finitely many extreme points $\lambda^k, k = 1, \ldots, K$. Hence,

$$v(s) = \max_{k=1,\ldots,K} \lambda^k \lceil s \rceil, \quad s \in \mathbb{R}^m,$$

and thus

$$Q(z) = \mathbb{E}_\omega \left[\max_{k=1,\ldots,K} \lambda^k \lceil \omega - z \rceil \right], \quad z \in \mathbb{R}^m. \tag{6}$$

Correspondingly, for every $\alpha \in \mathbb{R}^m$ the α-approximation Q_α can be written as

$$Q_\alpha(z) = \mathbb{E}_\omega \left[\max_{k=1,\ldots,K} \lambda^k \left(\lceil \omega \rceil_\alpha - z \right) \right], \quad z \in \mathbb{R}^m. \tag{7}$$

Now it is easy to observe that Q is the expectation of the pointwise maximum of finitely many round-up functions, so that Q is generally non-convex, whereas Q_α is a convex polyhedral function.

For TU integer recourse models no upper bound on $\|Q - Q_\alpha\|_\infty$ is available yet. Until recently the need for such an error bound appeared to be less urgent because Van der Vlerk [14] claims that in this case there exists $\alpha^* \in \mathbb{R}^m$ such that Q_{α^*} is the convex hull of Q, so that (under well-known assumptions) exact results are obtained. Indeed, in some exceptional cases the claim is valid. For example, if all random variables in the model are independent and uniformly distributed. However, in most cases the claim is not correct [9,10], so that an upper bound on $\|Q - Q_\alpha\|_\infty$ is required to guarantee the quality of the solutions of the approximating model.

3 Piecewise flattening of density functions without increasing total variation

The error bound for SIR models in (3) shows that the total variations of the densities of the random variables in the model are main determinants of the magnitude of the error $\|Q - Q_\alpha\|_\infty$. In this section we introduce several lemmas based on properties of the total variation of one-dimensional density functions. We use these lemmas extensively to solve a simplified one-dimensional bounding problem in Section 4, and to derive a bound for $\|Q - Q_\alpha\|_\infty$ for TU integer recourse models in Sections 5 and 6. In order to avoid technicalities, we only consider density functions f that are well-behaved in the following sense. (The obvious generalization to (in)dependent pdf on $\mathbb{R}^m$ is given in Sections 5 and 6).

Definition 2 Let $\mathcal{F}$ denote the set of one-dimensional probability density functions f of bounded variation that have finitely many discontinuity points on any bounded interval.

Remark 2 Note that for every $f \in \mathcal{F}$ there exists a left-continuous version $\hat{f} \in \mathcal{F}$ that is practically equivalent to f with $|\Delta|\hat{f} \leq |\Delta|f$.

The first lemma reads that the total variation does not increase when we *flatten* a density function on some bounded interval I in such a way that the probability of the event $\{\omega \in I\}$ does not change. The intuition behind this lemma is that a constant function has lower total variation than a varying one.

Lemma 1 *Let $f \in \mathcal{F}$ be given and let $I \subset \mathbb{R}$ denote a bounded interval with positive length $|I|$. Define $g \in \mathcal{F}$ as*

$$g(x) = \begin{cases} f(x), & x \notin I \\ K_I, & x \in I, \end{cases} \tag{8}$$

with $K_I := |I|^{-1} \int_I f(u)du$. Then $|\Delta|g \leq |\Delta|f$.

Proof Let $f \in \mathcal{F}$ be given and assume for the moment that I is open, so that $I = (a,b)$ for some $a < b$. Since $g(x) = f(x)$ for $x \notin (a,b)$, it follows that $|\Delta|g \leq |\Delta|f$ if and only if $|\Delta|g([a,b]) \leq |\Delta|f([a,b])$. Since g has the constant value K_I on the interval (a,b) it follows that

$$|\Delta|g([a,b]) = |K_I - f(a)| + |f(b) - K_I|.$$

In particular, if $\min\{f(a), f(b)\} \leq K_I \leq \max\{f(a), f(b)\}$ we have

$$|\Delta|g([a,b]) = |f(b) - f(a)| \leq |\Delta|f([a,b]).$$

For larger or smaller values of K_I we use that

$$|\Delta|f([a,b]) \geq |f(d) - f(a)| + |f(b) - f(d)| \quad \text{for all } d \in (a,b).$$

Note that there exists $d_1 \in (a,b)$ with $f(d_1) \leq K_I$. Otherwise, $\int_I f(u)du > \int_I K_I du = |I|K_I = \int_I f(u)du$ yields a contradiction. Similarly, there exists $d_2 \in (a,b)$ with $f(d_2) \geq K_I$.
Now suppose $K_I < \min\{f(a), f(b)\}$. Then

$$\begin{aligned} |\Delta|f([a,b]) &\geq |f(d_1) - f(a)| + |f(b) - f(d_1)| \\ &\geq |K_I - f(a)| + |f(b) - K_I| = |\Delta|g([a,b]), \end{aligned}$$

the latter inequality being true since $f(d_1) \leq K_I < \min\{f(a), f(b)\}$.
Analogously, if $K_I > \max\{f(a), f(b)\}$,

$$\begin{aligned} |\Delta|f([a,b]) &\geq |f(d_2) - f(a)| + |f(b) - f(d_2)| \\ &\geq |K_I - f(a)| + |f(b) - K_I| = |\Delta|g([a,b]). \end{aligned}$$

We conclude that $|\Delta|g([a,b]) \leq |\Delta|f([a,b])$ and thus $|\Delta|g \leq |\Delta|f$.

When I is not open, the proof is more technical but follows the same line of argument as above; therefore we omit this part of the proof. □

The next two lemmas use the result from Lemma 1 and are designed with deriving an upper bound for $\|Q - Q_\alpha\|_\infty$ in mind. Assuming the same properties as those of the functions involved, we show in Lemma 2 that flattening a density function leads to an expected value of zero for 'average-zero' functions, and in Lemma 3 we show that this operations can be carried out in such a way that the expected value of *piecewise constant* functions does not change.

Lemma 2 *Let φ be a bounded function with the property that $\int_I \varphi(x)dx = 0$ for some bounded interval I. Then for every $f \in \mathcal{F}$, there exists $g \in \mathcal{F}$ such that*

(i) $|\Delta|g \leq |\Delta|f$,

(ii) $g(x) = f(x)$, *for* $x \notin I$,

(iii) $\int_I \varphi(x)g(x)dx = 0$,

(iv) $\int \varphi(x)f(x)dx - \int \varphi(x)g(x)dx = \int_I \varphi(x)f(x)dx$.

For example, the pdf g defined in (8) satisfies these four properties.

Proof Let $f \in \mathcal{F}$ be given. Since φ is bounded it follows that $|\int \varphi(x)f(x)dx| < +\infty$. Define $g \in \mathcal{F}$ as in equation (8), hence by Lemma 1 properties (i) and (ii) follow. Because of (ii), $\int_{\mathbb{R}\setminus I} \varphi(x)g(x)dx = \int_{\mathbb{R}\setminus I} \varphi(x)f(x)dx$. Moreover, since g has constant value K_I on I, (iii) $\int_I \varphi(x)g(x)dx = K_I \int_I \varphi(x)dx = 0$, and (iv) follows immediately. □

Lemma 3 *Let $\varphi : \mathbb{R} \mapsto \mathbb{R}$ be a bounded* piecewise constant *function such that*

$$\varphi(x) := \sum_{j\in J} \varphi_j \mathbb{1}_{I_j}(x),$$

where $\mathbb{1}_I$ is the indicator function of interval I, $\{I_j\}_{j\in J}$ is a collection of disjoint bounded intervals of positive length such that $\cup_{j\in J} I_j = \mathbb{R}$, and $\varphi_j \in \mathbb{R}$, $j \in J$. Let V_φ denote the set of discontinuity points of φ. Assume that $|V_\varphi \cap I|$ is finite for any bounded interval I, then for every $f \in \mathcal{F}$ there exists a $g \in \mathcal{F}$ that is piecewise constant *with*

(i) $V_g \subseteq V_\varphi$,

(ii) $|\Delta|g \leq |\Delta|f$, *and*

(iii) $\int \varphi(x)g(x)dx = \int \varphi(x)f(x)dx$.

For example,

$$g(x) := |I_j|^{-1} \int_{I_j} f(u)du, \qquad \text{for } x \in I_j,\ j \in J \tag{9}$$

satisfies these properties.

Proof Let g be defined as in (9), so that g is a piecewise constant density function in $\mathcal{F}$ with (i) $V_g \subseteq V_\varphi$. Moreover, since $\int_{I_j} g(x)dx = \int_{I_j} f(x)dx$ for all $j \in J$, we have that

$$\begin{aligned}(iii) \int \varphi(x)f(x)dx &= \sum_{j\in J} \int_{I_j} \varphi(x)f(x)dx \\ &= \sum_{j\in J} \varphi_j \int_{I_j} f(x)dx \\ &= \sum_{j\in J} \varphi_j \int_{I_j} g(x)dx \\ &= \int \varphi(x)g(x)dx.\end{aligned}$$

By applying Lemma 1 repeatedly, we also have that (ii) $|\Delta|g \leq |\Delta|f$. □

Remark 3 Equivalently to $\int \varphi(x)g(x)dx = \int \varphi(x)f(x)dx$ we can write $\mathbb{E}_g[\varphi(\omega)] = \mathbb{E}_f[\varphi(\omega)]$, where $\mathbb{E}_g$ and $\mathbb{E}_f$ indicate that the expectation is with respect to g and f, respectively.

4 Uniform error bound for one-dimensional round-up functions

In the next sections we derive an error bound for the α-approximation Q_α of the TU integer recourse function Q. One of the main difficulties in calculating this error bound is that the maximizing dual vertices λ in (6) and (7) depend on ω, and are possibly different. If it were true that a deterministic $\hat{\lambda}$ exists such that

$$Q(z) = \mathbb{E}_\omega \left[\max_{k=1,\dots,K} \lambda^k \lceil \omega - z \rceil \right] \leq \mathbb{E}_\omega \left[\hat{\lambda} \lceil \omega - z \rceil \right]$$

and

$$Q_\alpha(z) = \mathbb{E}_\omega \left[\max_{k=1,\dots,K} \lambda^k (\lceil \omega \rceil_\alpha - z) \right] \geq \mathbb{E}_\omega \left[\hat{\lambda} (\lceil \omega \rceil_\alpha - z) \right],$$

then

$$Q(z) - Q_\alpha(z) \leq \mathbb{E}_\omega \left[\hat{\lambda} (\lceil \omega \rceil_z - \lceil \omega \rceil_\alpha) \right] = \sum_{i=1}^m \hat{\lambda}_i \mathbb{E}_{\omega_i} \left[\lceil \omega_i \rceil_{z_i} - \lceil \omega_i \rceil_{\alpha_i} \right],$$

so that we obtain an error bound if we derive a bound on each component of $\mathbb{E}_\omega[\lceil \omega \rceil_z - \lceil \omega \rceil_\alpha]$. In this section we analyze this simplified one-dimensional bounding problem. It can be solved by clever application of flattening of densities, using the special properties of the underlying difference function. Surprisingly, it appears to be true that the uniform upper bound of this hypothesized α-approximation is very useful for the TU model, to be discussed in the next section. As we will show then, a suitable relaxation of the set of dual vertices λ to a set with deterministic pointwise supremum λ^* is possible, and together with suitable flattening of the densities involved an error bound will be derived.

Definition 3 (Difference function) For every $\alpha \in \mathbb{R}$, $z \in \mathbb{R}$, define the *difference function* $\varphi_{\alpha,z}$ as

$$\varphi_{\alpha,z}(x) := \lceil x \rceil_z - \lceil x \rceil_\alpha = \lceil x - z \rceil + z - \lceil x - \alpha \rceil - \alpha, \quad x \in \mathbb{R}.$$

Moreover, for every $\alpha \in \mathbb{R}, z \in \mathbb{R}$, define the *expected difference function* $D_{\alpha,z} : \mathcal{F} \mapsto \mathbb{R}$ as

$$D_{\alpha,z}(f) := \mathbb{E}_f[\varphi_{\alpha,z}(\omega)], \quad f \in \mathcal{F}.$$

Remark 4 For fixed $\alpha \in \mathbb{R}$ and $f \in \mathcal{F}$, the expected difference function $D_{\alpha,z}(f)$ can be interpreted as the difference between the round-up function $R(z) := \mathbb{E}_\omega[\lceil \omega - z \rceil]$, $z \in \mathbb{R}$, and its α-approximation $\mathbb{E}_\omega[(\lceil \omega \rceil_\alpha - z)]$, where the expectations are with respect to the pdf f.

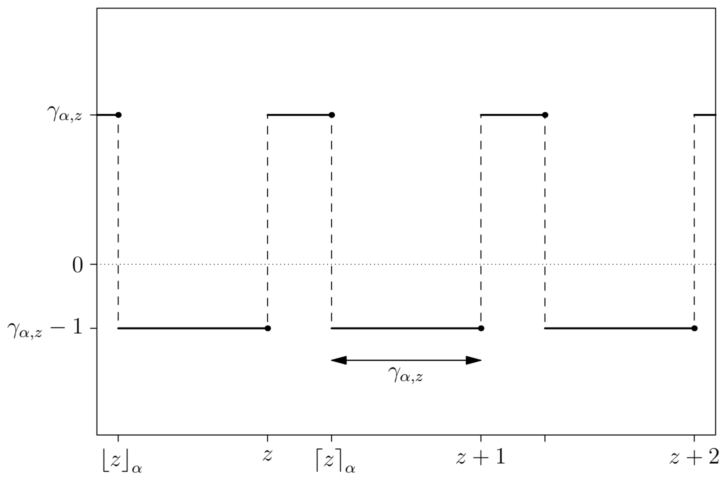

Fig. 1 The difference function $\varphi_{\alpha,z}$ from Definition 3.

The properties of the difference function $\varphi_{\alpha,z}$ are summarized in Lemma 4, see also Figure 1.

Lemma 4 (Properties of the difference function) *Consider the difference function $\varphi_{\alpha,z}(x) := \lceil x \rceil_z - \lceil x \rceil_\alpha$, $x \in \mathbb{R}$.*

(i) $\varphi_{\alpha,z}$ *is periodic in* x, α, *and* z *with period 1, and moreover* $\varphi_{\alpha,z}(x) = -\varphi_{z,\alpha}(x)$.

(ii) If $\alpha - z \in \mathbb{Z}$ *then* $\varphi_{\alpha,z} \equiv 0$.

(iii) If $\alpha - z \notin \mathbb{Z}$ *then* $\varphi_{\alpha,z}$ *is a two-valued function*

$$\varphi_{\alpha,z}(x) = \begin{cases} \gamma_{\alpha,z}, & x \in \cup_{l\in\mathbb{Z}}(z+l, \lceil z \rceil_\alpha + l], \\ \gamma_{\alpha,z} - 1, & x \in \cup_{l\in\mathbb{Z}}(\lfloor z \rfloor_\alpha + l, z + l], \end{cases} \tag{10}$$

with

$$\gamma_{\alpha,z} := z - \lfloor z \rfloor_\alpha = z + 1 - \lceil z \rceil_\alpha \in (0,1).$$

Thus, $\varphi_{\alpha,z}$ *has jumps of size* $+1$ *on* $z+\mathbb{Z}$ *and jumps of size* -1 *on* $\alpha+\mathbb{Z}$*, and it is left-continuous.*

(iv) $\int_I \varphi_{\alpha,z}(x)dx = 0$ *for any interval* I *of length* $|I| = 1$.

Proof Properties (i) and (ii) are obvious. (iii) Since $\lceil x - y \rceil + y$ is a piecewise constant (left-continuous) function with jumps of size $+1$ on $y + \mathbb{Z}$, it follows that $\varphi_{\alpha,z}$ is piecewise constant (left-continuous) with jumps of size $+1$ on $z+\mathbb{Z}$ and jumps of size -1 on $\alpha + \mathbb{Z}$.
Note that for $x \in (z, \lceil z \rceil_\alpha]$,

$$\varphi_{\alpha,z}(x) = z + 1 - \lceil z - \alpha \rceil - \alpha = z + 1 - \lceil z \rceil_\alpha = z - \lfloor z \rfloor_\alpha = \gamma_{\alpha,z} \in (0,1).$$

Since $\varphi_{\alpha,z}$ has jumps of size -1 on $\alpha + \mathbb{Z}$, it follows that

$$\varphi_{\alpha,z}(x) = \gamma_{\alpha,z} - 1, \qquad \text{for } x \in (\lceil z \rceil_\alpha, z + 1].$$

Since $\varphi_{\alpha,z}$ is periodic with period 1, equation (10) holds. Moreover, we have

$$\int_{\lfloor z \rfloor_\alpha}^{\lceil z \rceil_\alpha} \varphi_{\alpha,z}(x)dx = \int_{\lfloor z \rfloor_\alpha}^{z} \varphi_{\alpha,z}(x)dx + \int_{z}^{\lceil z \rceil_\alpha} \varphi_{\alpha,z}(x)dx = 0,$$

since

$$\int_{\lfloor z \rfloor_\alpha}^{z} \varphi_{\alpha,z}(x)dx = (z - \lfloor z \rfloor_\alpha)(\gamma_{\alpha,z} - 1) = -\gamma_{\alpha,z}(1 - \gamma_{\alpha,z}) \tag{11}$$

and

$$\int_{z}^{\lceil z \rceil_\alpha} \varphi_{\alpha,z}(x)dx = (\lceil z \rceil_\alpha - z)\gamma_{\alpha,z} = (1 - \gamma_{\alpha,z})\gamma_{\alpha,z}. \tag{12}$$

From the periodicity of $\varphi_{\alpha,z}$ it now follows that (iv) $\int_I \varphi_{\alpha,z}(x)dx = 0$ for any interval I of length $|I| = 1$. $\square$

The following properties of the expected difference function $D_{\alpha,z}$ follow directly from Lemma 4.

Corollary 1 *For every $f \in \mathcal{F}$,*

(i) $D_{\alpha,z}(f)$ *is periodic in both* α *and* z *with period 1,*

(ii) $D_{\alpha,z}(f) = -D_{z,\alpha}(f)$, *and*

(iii) $D_{\alpha,z}(f) = 0$ *if* $\alpha - z \in \mathbb{Z}$.

After these technical preparations we are ready to derive an upper bound for $|D_{\alpha,z}(f)|$. Obviously, for any given $f_0 \in \mathcal{F}$ and any $\alpha \in \mathbb{R}$ the sharpest upper bound is

$$\mathcal{M}(\alpha, f_0) := \sup_{z\in\mathbb{R}} |D_{\alpha,z}(f_0)|. \tag{13}$$

However, it is practically impossible to calculate this bound. Surprisingly, a kind of worst-case analysis appears to be very helpful. Instead of considering f_0 which has $|\Delta| f_0 = B_0$, we will solve, for all $B > 0$, the optimization problem

$$M(B) := \sup_{\alpha\in\mathbb{R}} \sup_{f\in\mathcal{F}} \Big\{ \mathcal{M}(\alpha, f) : |\Delta| f \le B \Big\},$$

so that $M(B_0)$ is an upper bound for $\mathcal{M}(\alpha, f_0)$. This key result is contained in Theorem 1, concluding this section.

We first explain why the worst-case approach works. By interchanging supremizations and using $D_{\alpha,z}(f) = -D_{z,\alpha}(f)$, it follows that

$$\begin{aligned} M(B) &= \sup_{\alpha\in\mathbb{R}} \sup_{z\in\mathbb{R}} \sup_{f\in\mathcal{F}} \Big\{ |D_{\alpha,z}(f)| : |\Delta| f \le B \Big\} \\ &= \sup_{\alpha\in\mathbb{R}} \sup_{z\in\mathbb{R}} \sup_{f\in\mathcal{F}} \Big\{ D_{\alpha,z}(f) : |\Delta| f \le B \Big\}. \end{aligned} \tag{14}$$

We will show that the inner supremization,

$$(\mathcal{P}) \qquad \sup_{f\in\mathcal{F}} \Big\{ D_{\alpha,z}(f) : |\Delta| f \le B \Big\},$$

with fixed α and z, can be solved explicitly, using the tools of Section 3.

Proposition 1 *Let $\alpha, z \in \mathbb{R}$ be given. Then, for every $B > 0$,*

$$\sup_{f\in\mathcal{F}} \Big\{ D_{\alpha,z}(f) : |\Delta| f \le B \Big\} = \min \Big\{ \gamma_{\alpha,z}, \gamma_{\alpha,z}(1 - \gamma_{\alpha,z}) \frac{B}{2} \Big\}, \tag{15}$$

with $\gamma_{\alpha,z} := z - \lfloor z \rfloor_\alpha$.

Proof If $\alpha - z \in \mathbb{Z}$, so that $\gamma_{\alpha,z} = 0$, then Corollary 1 (iii) shows that $D_{\alpha,z}(f) = 0$ for all $f \in \mathcal{F}$ so that $\sup_{f\in\mathcal{F}} \Big\{ D_{\alpha,z}(f) : |\Delta| f \le B \Big\} = 0$ and thus (15) holds.

If $\alpha - z \notin \mathbb{Z}$, then the difference function $\varphi_{\alpha,z}$ is piecewise constant with $V_{\varphi_{\alpha,z}} = (\alpha + \mathbb{Z}) \cup (z + \mathbb{Z})$ so that it satisfies the conditions of Lemma 3. Application of this lemma shows, that for every feasible f of maximization problem $\mathcal{P}$ there exists a *piecewise constant* feasible solution g with the same objective value, and with $V_g \subset V_{\varphi_{\alpha,z}}$. Hence, we can (and will) restrict the feasible region of $(\mathcal{P})$ to piecewise constant density functions f with $V_f \subset (\alpha+\mathbb{Z})\cup(z+\mathbb{Z})$. We will denote its function values to the left of $z+l$ by f_l^-, and to the right of $z+l$ by f_l^+, that is

$$f(x) = \begin{cases} f_l^-, & \text{for } x \in (\lfloor z \rfloor_\alpha + l, z+l], \quad l \in \mathbb{Z} \\ f_l^+, & \text{for } x \in (z+l, \lceil z \rceil_\alpha + l], \quad l \in \mathbb{Z}. \end{cases}$$

Consider such feasible $f \in \mathcal{F}$. We will derive necessary optimality conditions on its function values by applying Lemma 2 with $\varphi = \varphi_{\alpha,z}$ and I arbitrary with $|I| = 1$. Lemma 4 (iv) shows that the conditions of Lemma 2 are satisfied. Lemma 2 (i, iv) shows that a feasible g exists such that $D_{\alpha,z}(f) - D_{\alpha,z}(g) = \int_I \varphi_{\alpha,z}(x)f(x)dx$. If the right-hand side happens to be negative, f cannot be optimal for $(\mathcal{P})$ since g has a better objective value. Hence, for each interval I with $|I| = 1$ we have the following necessary optimality condition for f in $(\mathcal{P})$:

$$\int_I \varphi_{\alpha,z}(x)f(x)dx \geq 0.$$

In particular, for $I = (z+l-1, z+l]$ and $I = (\lfloor z \rfloor_\alpha + l, \lceil z \rceil_\alpha + l]$, $l \in \mathbb{Z}$, it can be derived from (11) and (12) that

$$\int_{z+l-1}^{z+l} \varphi_{\alpha,z}(x)f(x)dx = \gamma_{\alpha,z}(1-\gamma_{\alpha,z})\{f_{l-1}^+ - f_l^-\},$$

and

$$\int_{\lfloor z \rfloor_\alpha + l}^{\lceil z \rceil_\alpha + l} \varphi_{\alpha,z}(x)f(x)dx = \gamma_{\alpha,z}(1-\gamma_{\alpha,z})\{f_l^+ - f_l^-\},$$

yielding the optimality conditions

$$f_{l-1}^+ \geq f_l^-, \quad l \in \mathbb{Z},$$

and

$$f_l^+ \geq f_l^-, \quad l \in \mathbb{Z}.$$

Under these restrictions f is a piecewise constant density function whose value alternatingly increases and decreases. For such density functions the total variation can be expressed as $|\Delta|f = 2\sum_{l\in\mathbb{Z}}\{f_l^+ - f_l^-\}$, i.e., as the sum of its total increase and total decrease. Moreover, using (11), (12), and the periodicity of $\varphi_{\alpha,z}$ we have that

$$\begin{aligned} D_{\alpha,z}(f) &= \int \varphi_{\alpha,z}(x)f(x)dx \\ &= \sum_{l\in\mathbb{Z}} \left\{ f_l^- \int_{\lfloor z \rfloor_\alpha + l}^{z+l} \varphi_{\alpha,z}(x)dx + f_l^+ \int_{z+l}^{\lceil z \rceil_\alpha + l} \varphi_{\alpha,z}(x)dx \right\} \\ &= \gamma_{\alpha,z}(1-\gamma_{\alpha,z}) \sum_{l\in\mathbb{Z}} \left\{ f_l^+ - f_l^- \right\}. \end{aligned}$$

Hence, problem $(\mathcal{P})$ reduces to the optimization problem

$$\begin{aligned}
\sup_{f_l^+, f_l^-} \quad & D_{\alpha,z}(f) = \gamma_{\alpha,z}(1-\gamma_{\alpha,z}) \sum_{l\in\mathbb{Z}} \left\{ f_l^+ - f_l^- \right\} \\
\text{s.t.} \quad & \sum_{l\in\mathbb{Z}} \left\{ (1-\gamma_{\alpha,z}) f_l^+ + \gamma_{\alpha,z} f_l^- \right\} = 1 && (16) \\
& \sum_{l\in\mathbb{Z}} \left\{ f_l^+ - f_l^- \right\} \leq \frac{B}{2} && (17) \\
& f_l^+ \geq f_l^-, \quad f_{l-1}^+ \geq f_l^-, \quad l \in \mathbb{Z} && (18) \\
& f_l^+ \geq 0, \quad f_l^- \geq 0, \qquad l \in \mathbb{Z} && (19)
\end{aligned}$$

Here, (16), (19), and (17) ensure that f is a pdf with $|\Delta| f \leq B$, whereas the inequalities in (18) represent the necessary optimality conditions derived above. Notice that the variables f_l^+ have a positive coefficient in the objective, and f_l^- a negative one.

We solve this reduced version of $(\mathcal{P})$ by providing an upper bound which we subsequently prove to be tight. On the one hand (17) implies that

$$D_{\alpha,z}(f) \leq \gamma_{\alpha,z}(1-\gamma_{\alpha,z}) \frac{B}{2}, \tag{20}$$

and on the other hand, since (16) is equivalent to

$$(1-\gamma_{\alpha,z}) \sum_{l\in\mathbb{Z}} \{ f_l^+ - f_l^- \} = 1 - \sum_{l\in\mathbb{Z}} f_l^-,$$

we have

$$\begin{aligned}
D_{\alpha,z}(f) &= \gamma_{\alpha,z}(1-\gamma_{\alpha,z}) \sum_{l\in\mathbb{Z}} \{ f_l^+ - f_l^- \} \\
&= \gamma_{\alpha,z} \left(1 - \sum_{l\in\mathbb{Z}} f_l^- \right) \\
&\leq \gamma_{\alpha,z}, && (21)
\end{aligned}$$

since $\sum_{l\in\mathbb{Z}} f_l^- \geq 0$. Combining the upper bounds in (20) and (21) yields, for every $f \in \mathcal{F}$ with $|\Delta| f \leq B$,

$$\begin{aligned}
D_{\alpha,z}(f) &\leq \min\{\gamma_{\alpha,z}, \gamma_{\alpha,z}(1-\gamma_{\alpha,z})B/2\} \\
&= \begin{cases} \gamma_{\alpha,z}, & \text{if } \gamma_{\alpha,z} \leq 1 - 2/B, \\ \gamma_{\alpha,z}(1-\gamma_{\alpha,z})B/2, & \text{if } \gamma_{\alpha,z} \geq 1 - 2/B. \end{cases}
\end{aligned}$$

Consider first the case $0 < \gamma_{\alpha,z} \leq 1 - 2/B$ (i.e. $(1-\gamma_{\alpha,z})^{-1} \leq B/2$). Then the density $\hat{f}$ with

$$\hat{f}_0^- = 0,\ \hat{f}_0^+ = c,\ \hat{f}_l^- = \hat{f}_l^+ = 0 \quad \text{for all } l \in \mathbb{Z}\backslash\{0\}$$

satisfies all constraints (16) – (19) if $c := (1-\gamma_{\alpha,z})^{-1}$, and the objective value $D_{\alpha,z}(\hat{f})$ equals $\gamma_{\alpha,z}$, indeed.

Consider next the case $1 - 2/B < \gamma_{\alpha,z} < 1$ (so that $(1-\gamma_{\alpha,z})B/2 < 1$). Then the density $\bar{f}$ with

$$\begin{aligned} \bar{f}_0^- = 0,\ \bar{f}_0^+ = B/2, \bar{f}_l^- = \bar{f}_l^+ &= c \quad l = 1, \dots, k \\ \bar{f}_l^- = \bar{f}_l^+ &= 0 \quad l < 0,\ l > k \end{aligned}$$

satisfies all constraints (16) – (19) if

$$\begin{aligned} (1-\gamma_{\alpha,z})B/2 + kc = 1 & \qquad \text{(from (16))} \\ 0 \le c \le B/2 & \qquad \text{(from } 0 \le \bar{f}_1^- \le \bar{f}_0^+\text{)} \end{aligned}$$

and these are satisfied by $k = k^*$, $c = c^*$ given by

$$k^* := \min_{k \in \mathbb{Z}} \{k : (1-\gamma_{\alpha,z})B/2 + kB/2 \ge 1\} = \lceil \gamma_{\alpha,z} - (1 - 2/B) \rceil \tag{22}$$

$$c^* := (1 - (1-\gamma_{\alpha,z})B/2)/k^*. \tag{23}$$

The objective value $D_{\alpha,z}(\bar{f})$ equals $\gamma_{\alpha,z}(1-\gamma_{\alpha,z})B/2$, indeed. □

It is interesting to picture the optimal densities $\hat{f}$ and $\bar{f}$ from the proof of Proposition 1 because for these densities the error of the α-approximation is largest. Obviously, the shape of such an optimal density will depend on the value of B.

For large values of B, the constraint on the total variation of f is not very restrictive. Therefore, it is not hard to imagine that (since $\varphi_{\alpha,z}$ is two-valued with maximum value $\gamma_{\alpha,z}$) it might be possible to attain the upper bound $\gamma_{\alpha,z}$ by setting $f(x) > 0$ if and only if $\varphi_{\alpha,z}(x) = \gamma_{\alpha,z} > 0$. It turns out that this is indeed possible if $\gamma_{\alpha,z} \le 1 - 2/B$. For example, the pdf $\hat{f}$ defined as

$$\hat{f}(x) = \begin{cases} (1-\gamma_{\alpha,z})^{-1}, & z < x \le \lceil z \rceil_\alpha \\ 0, & \text{otherwise,} \end{cases} \tag{24}$$

has objective value $D_{\alpha,z}(\hat{f}) = \gamma_{\alpha,z}$.

For smaller values of B for which $1 - 2/B < \gamma_{\alpha,z} < 1$, the pdf $\hat{f}$ is infeasible because it violates the total variation constraint. In fact, any pdf f with $D_{\alpha,z}(f) = \gamma_{\alpha,z}$ now violates this constraint, so that intuitively any optimal pdf f must satisfy $|\Delta| f = B$. An example of such an optimal density is given by the pdf $\bar{f}$ (see Figure 2) defined as

$$\bar{f}(x) = \begin{cases} B/2, & x \in (z, \lceil z \rceil_\alpha] \\ c^*, & x \in (\lceil z \rceil_\alpha, \lceil z \rceil_\alpha + k^*] \\ 0, & \text{otherwise,} \end{cases} \tag{25}$$

with k^* and c^* defined in (22) and (23), respectively. Indeed, it can be shown that any pdf f that is piecewise constant with $V_f \subset (\alpha + \mathbb{Z}) \cup (z + \mathbb{Z})$ satisfying

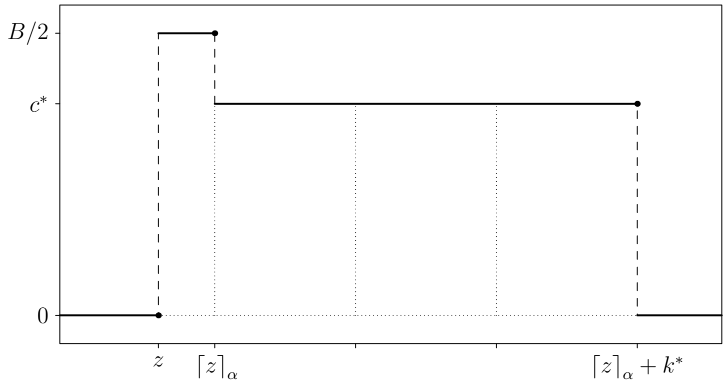

Fig. 2 The pdf $\bar{f}$ defined in (25) with $k^* = 3$.

(16), (18), (19), and $|\Delta|f = B$ is optimal with objective value $D_{\alpha,z}(f) = D_{\alpha,z}(\bar{f}) = \gamma_{\alpha,z}(1 - \gamma_{\alpha,z})B/2$.

Now that we have solved the inner optimization problem ($\mathcal{P}$) explicitly, it is easy to find an upper bound for $\mathcal{M}(\alpha, f)$.

Theorem 1 (Error bound for the expected difference function) *For every $\alpha \in \mathbb{R}$ and every random variable ω with pdf $f \in \mathcal{F}$,*

$$\mathcal{M}(\alpha, f) := \sup_{z \in \mathbb{R}} |D_{\alpha,z}(f)| \leq h(|\Delta|f),$$

where $h : \mathbb{R}_{++} \mapsto \mathbb{R}$ is given by

$$h(x) = \begin{cases} x/8, & 0 < x \leq 4, \\ 1 - 2/x, & x \geq 4. \end{cases} \tag{26}$$

Proof Let $f_0 \in \mathcal{F}$ with $|\Delta|f_0 = B_0$ be given. Then, $M(B_0)$ with M as in (14) is an upper bound of $\mathcal{M}(\alpha, f_0)$. Using Proposition 1, we have that

$$M(B_0) = \sup_{\alpha \in \mathbb{R}} \sup_{z \in \mathbb{R}} \min \left\{ \gamma_{\alpha,z}, \gamma_{\alpha,z}(1 - \gamma_{\alpha,z}) \frac{B_0}{2} \right\},$$

with $\gamma_{\alpha,z} := z - \lfloor z \rfloor_\alpha \in [0, 1)$. Hence, it follows that

$$M(B_0) = \sup_{\gamma \in [0,1)} \min \left\{ \gamma, \gamma(1 - \gamma) \frac{B_0}{2} \right\}.$$

In this optimization problem we have to maximize the minimum of a linear and a quadratic function over the domain $[0, 1)$. Elementary analysis shows that the optimal solution is given by $\gamma_{B_0} := \max\{1/2, 1 - 2/B_0\}$, whereas the optimal value is equal to $h(B_0)$, where h is defined in (26). $\square$

5 TU integer recourse models with independent random variables

Now we have set the stage for the analysis of TU integer recourse models. To avoid obscuring technicalities we first assume that the components of the m-dimensional random right-hand side vector ω are independently distributed and that the joint density function f of ω is contained in $\mathcal{F}^m$, to be defined below. We will deal with dependent distributions in the next section.

Definition 4 Let $\mathcal{F}^m$ denote the set of m-dimensional joint density functions f whose marginal densities $f_i, i = 1, \dots, m$, are contained in $\mathcal{F}$, and for which

$$f(x) = \prod_{i=1}^{m} f_i(x_i), \quad x \in \mathbb{R}^m.$$

We will derive an error bound for the α-approximation Q_α of the TU integer recourse function Q given by (7) and (6), respectively. Similar as for the expected difference function in Section 4, for almost any given $f \in \mathcal{F}^m$ with $|\Delta| f_i = B_i$ and $\alpha \in \mathbb{R}^m$, direct calculation of the sharpest upper bound

$$\mathcal{N}(\alpha, f) := \sup_{z \in \mathbb{R}^m} |Q(z) - Q_\alpha(z)|$$

is too demanding. As already mentioned, one of the main difficulties in calculating this bound is that the maximizing dual vertices λ in (6) and (7) depend on ω, and are possibly different. In order to overcome this difficulty we relax the set of possible dual vertices and use a worst-case analysis over this relaxed set. As we will see, this approach, combined with the analysis of the one-dimensional expected difference function, yields the desired upper bound.

Consider, therefore, the TU integer expected value function Q and pick for every $z \in \mathbb{R}^m$, a function $\lambda_Q^z : \mathbb{R}^m \mapsto \mathbb{R}^m$ such that

$$\lambda_Q^z(x) \in \operatorname*{argmax}_{k=1,\dots,K} \lambda^k \lceil x - z \rceil, \quad x \in \mathbb{R}^m, \tag{27}$$

and λ_Q^z is constant on

$$C_z^l := \prod_{i=1}^{m} C_{z_i}^{l_i} := \prod_{i=1}^{m} (z_i + l_i - 1, z_i + l_i]$$

for every $l \in \mathbb{Z}^m$. This is indeed possible since $\lceil x - z \rceil$ is constant on C_z^l. Analogously, associated with Q_α, pick for every $\alpha \in \mathbb{R}^m$ and $z \in \mathbb{R}^m$,

$$\lambda_{Q_\alpha}^z(x) \in \operatorname*{argmax}_{k=1,\dots,K} \lambda^k (\lceil x \rceil_\alpha - z), \quad x \in \mathbb{R}^m,$$

such that $\lambda_{Q_\alpha}^z$ is constant on C_α^l for every $l \in \mathbb{Z}^m$. Now we can rewrite Q and Q_α as $Q(z) = \mathbb{E}_\omega[\lambda_Q^z(\omega) \lceil \omega - z \rceil]$ and $Q_\alpha(z) = \mathbb{E}_\omega[\lambda_{Q_\alpha}^z(\omega)(\lceil \omega \rceil_\alpha - z)]$, respectively.

Note that λ_Q^z and $\lambda_{Q_\alpha}^z$ have three important properties in common. First of all, both functions are nonnegative. Second, both functions are bounded by $\lambda^* \in \mathbb{R}^m$ defined as

$$\lambda_i^* := \max_{k=1,\ldots,K} \lambda_i^k, \tag{28}$$

and third, for both functions there exists $\beta \in \mathbb{R}^m$ such that the function is constant on C_β^l for every $l \in \mathbb{Z}^m$. These three properties are paramount to obtain an upper bound for $\mathcal{N}(\alpha, f)$ as we show now.

Definition 5 Let Λ^m denote the set of functions $\lambda : \mathbb{R}^m \mapsto \mathbb{R}^m$ for which

(i) $0 \leq \lambda(x) \leq \lambda^*$, for every $x \in \mathbb{R}^m$, and

(ii) there exists $\beta \in \mathbb{R}^m$ such that λ is constant on C_β^l for every $l \in \mathbb{Z}^m$.

Definition 6 For every $\alpha \in \mathbb{R}^m, z \in \mathbb{R}^m$, define $G_{\alpha,z} : \Lambda^m \times \mathcal{F}^m \mapsto \mathbb{R}$ as

$$G_{\alpha,z}(\lambda, f) := \mathbb{E}_f\Big[\lambda(\omega)\Big(\lceil \omega \rceil_z - \lceil \omega \rceil_\alpha \Big)\Big],$$

where $\lambda \in \Lambda^m$ and $f \in \mathcal{F}^m$.

Lemma 5 *For every $\hat{\alpha} \in \mathbb{R}^m$ and every $f \in \mathcal{F}^m$,*

$$\mathcal{N}(\hat{\alpha}, f) \leq \mathcal{N}^*(f) := \sup_{\alpha \in \mathbb{R}^m} \sup_{z \in \mathbb{R}^m} \sup_{\lambda \in \Lambda^m} G_{\alpha,z}(\lambda, f).$$

Proof Let $\hat{\alpha} \in \mathbb{R}^m$ and $f \in \mathcal{F}^m$ be given. We will show that for every $z \in \mathbb{R}^m$,

$$Q(z) - Q_{\hat{\alpha}}(z) \leq \sup_{\lambda \in \Lambda^m} G_{\hat{\alpha},z}(\lambda, f),$$

and

$$Q_{\hat{\alpha}}(z) - Q(z) \leq \sup_{\lambda \in \Lambda^m} G_{z,\hat{\alpha}}(\lambda, f),$$

implying that

$$\sup_{z \in \mathbb{R}^m} |Q(z) - Q_{\hat{\alpha}}(z)| \leq \sup_{\alpha \in \mathbb{R}^m} \sup_{z \in \mathbb{R}^m} \sup_{\lambda \in \Lambda^m} G_{\alpha,z}(\lambda, f)$$

as postulated.

To prove the first inequality, let $z \in \mathbb{R}^m$ be given and consider the function λ_Q^z as defined in (27). Note that $\lambda_Q^z(x)$ is a maximizer of $\max_{k=1,\ldots,K} \lambda^k \lceil x - z \rceil$ for every $x \in \mathbb{R}^m$, but not necessarily of $\max_{k=1,\ldots,K} \lambda^k(\lceil x \rceil_{\hat{\alpha}} - z)$. Thus,

$$Q(z) - Q_{\hat{\alpha}}(z) \leq \mathbb{E}_\omega\Big[\lambda_Q^z\Big\{ \lceil \omega \rceil_z - \lceil \omega \rceil_{\hat{\alpha}} \Big\}\Big] = G_{\hat{\alpha},z}(\lambda_Q^z, f).$$

Since $\lambda_Q^z \in \Lambda^m$, the first inequality follows. Analogously, the second inequality follows from

$$Q_{\hat{\alpha}}(z) - Q(z) \leq \mathbb{E}_\omega\Big[\lambda_{Q_{\hat{\alpha}}}^z\{\lceil \omega \rceil_{\hat{\alpha}} - \lceil \omega \rceil_z\}\Big] = G_{z,\hat{\alpha}}(\lambda_{Q_{\hat{\alpha}}}^z, f). \qquad \square$$

The final step in our analysis comprises a similar worst-case analysis as carried out for the one-dimensional case in the previous section. We consider, for all $B \in \mathbb{R}^m_{++}$, the optimization problem

$$N(B) := \sup_{f \in \mathcal{F}^m} \left\{ \mathcal{N}^*(f) : |\Delta| f_i \leq B_i,\ i = 1, \dots, m \right\} \tag{29}$$

$$= \sup_{\alpha \in \mathbb{R}^m} \sup_{z \in \mathbb{R}^m} \sup_{f \in \mathcal{F}^m} \sup_{\lambda \in \Lambda^m} \left\{ G_{\alpha,z}(\lambda, f) : |\Delta| f_i \leq B_i,\ i = 1, \dots, m \right\}.$$

The following proposition allows us to reduce the problem to one involving the constant function $\lambda \equiv \lambda^*$, with λ^* defined in (28).

Proposition 2 *For every $\alpha \in \mathbb{R}^m, z \in \mathbb{R}^m, \lambda \in \Lambda^m$, and $f \in \mathcal{F}^m$, there exists $g \in \mathcal{F}^m$ with $|\Delta| g_i \leq |\Delta| f_i, i = 1, \dots, m$, such that $G_{\alpha,z}(\lambda, f) \leq G_{\alpha,z}(\lambda, g) \leq G_{\alpha,z}(\lambda^*, g)$.*

Proof Let $\alpha \in \mathbb{R}^m, z \in \mathbb{R}^m, \lambda \in \Lambda^m$, and $f \in \mathcal{F}^m$ be given with λ constant on every C^l_β for some $\beta \in \mathbb{R}^m$. Observe that

$$\begin{aligned} G_{\alpha,z}(\lambda, f) :=& \mathbb{E}_\omega \left[\lambda(\omega) \Big(\lceil \omega \rceil_z - \lceil \omega \rceil_\alpha \Big) \right] \\ =& \mathbb{E}_\omega \left[\sum_{i=1}^m \lambda_i(\omega) \varphi_{\alpha_i, z_i}(\omega_i) \right] \\ =& \sum_{i=1}^m \int_{\mathbb{R}^m} \lambda_i(x) \varphi_{\alpha_i, z_i}(x_i) f(x) dx, \end{aligned}$$

where φ_{α_i,z_i} is the one-dimensional difference function introduced in Definition 3. Since λ is constant on C^l_β for every l, we can calculate the expected value on each C^l_β separately:

$$\begin{aligned} G_{\alpha,z}(\lambda, f) &= \sum_{i=1}^m \sum_{l \in \mathbb{Z}^m} \int_{C^l_\beta} \lambda_i(x) \varphi_{\alpha_i, z_i}(x_i) f(x) dx \\ &= \sum_{i=1}^m \sum_{l \in \mathbb{Z}^m} \lambda_i(l + \beta) \int_{C^l_\beta} \varphi_{\alpha_i, z_i}(x_i) f(x) dx. \end{aligned}$$

Moreover, since $C^l_\beta = \prod_{j=1}^m C^{l_j}_{\beta_j}$ and $f(x) = \prod_{j=1}^m f_j(x_j)$, we obtain

$$\int_{C^l_\beta} \varphi_{\alpha_i, z_i}(x_i) f(x) dx = \left(\int_{C^{l_i}_{\beta_i}} \varphi_{\alpha_i, z_i}(x_i) f_i(x_i) dx_i \right) \prod_{j \neq i} \int_{C^{l_j}_{\beta_j}} f_j(x_j) dx_j.$$

Writing $l_{(i)} := (l_1, \dots, l_{i-1}, l_{i+1}, \dots, l_m)$, we replace $\sum_{l \in \mathbb{Z}^m}$ by $\sum_{l_i \in \mathbb{Z}} \sum_{l_{(i)} \in \mathbb{Z}^{m-1}}$ and get

$$G_{\alpha,z}(\lambda, f) = \sum_{i=1}^m \sum_{l_i \in \mathbb{Z}} \psi_{\alpha,z,\lambda,f}(i, l_i) \int_{C^{l_i}_{\beta_i}} \varphi_{\alpha_i, z_i}(x_i) f_i(x_i) dx_i \tag{30}$$

with

$$\psi_{\alpha,z,\lambda,f}(i,l_i) := \sum_{l_{(i)} \in \mathbb{Z}^{m-1}} \lambda_i(l+\beta) \prod_{j \neq i} \int_{C_{\beta_j}^{l_j}} f_j(x_j)dx_j. \tag{31}$$

Observe that $\psi_{\alpha,z,\lambda,f}(i,l_i) \geq 0$ for every $i = 1,\dots,m$, $l_i \in \mathbb{Z}$. Thus, if we adapt f such that the integrals in (30) and (31) do not decrease, then an upper bound for $G_{\alpha,z}(\lambda,f)$ is obtained. To this end, we construct the joint density function $g \in \mathcal{F}^m$ as follows. Let

$$g(x) := \prod_{i=1}^{m} g_i(x_i), \quad x \in \mathbb{R}^m,$$

where for every $i = 1,\dots,m$, the marginal density function g_i is a special flattened version of f_i. To be specific, the function f_i is only flattened over those intervals $C_{\beta_i}^{l_i}$ for which $\int_{C_{\beta_i}^{l_i}} \varphi_{\alpha_i,z_i}(u)f_i(u)du < 0$. That is, for every $l_i \in \mathbb{Z}$, and $x_i \in C_{\beta_i}^{l_i}$,

$$g_i(x_i) := \begin{cases} f_i(x_i), & \text{if } \int_{C_{\beta_i}^{l_i}} \varphi_{\alpha_i,z_i}(u)f_i(u)du \geq 0, \quad (32\text{a}) \\ \int_{C_{\beta_i}^{l_i}} f_i(u)du, & \text{otherwise.} \quad (32\text{b}) \end{cases}$$

Obviously, because of Lemma 1, $|\Delta|g_i \leq |\Delta|f_i$, $i = 1,\dots,m$. In order to show that $G_{\alpha,z}(\lambda,f) \leq G_{\alpha,z}(\lambda,g) \leq G_{\alpha,z}(\lambda^*,g)$, notice that for every $l_i \in \mathbb{Z}$ and every $i = 1,\dots,m$,

(i) $\int_{C_{\beta_i}^{l_i}} g_i(u)du = \int_{C_{\beta_i}^{l_i}} f_i(u)du$

(ii) $\int_{C_{\beta_i}^{l_i}} \varphi_{\alpha_i,z_i}(u)g_i(u)du \geq \int_{C_{\beta_i}^{l_i}} \varphi_{\alpha_i,z_i}(u)f_i(u)du$

(iii) $\int_{C_{\beta_i}^{l_i}} \varphi_{\alpha_i,z_i}(u)g_i(u)du \geq 0$

These properties follow directly from the construction. Indeed, if case (32a) holds, nothing has to be shown. If case (32b) applies, (i) is obvious and

$$0 = \int_{C_{\beta_i}^{l_i}} \varphi_{\alpha_i,z_i}(u)g_i(u)du > \int_{C_{\beta_i}^{l_i}} \varphi_{\alpha_i,z_i}(u)f_i(u)du,$$

where the equality follows from Lemma 2 (iii) using $|C_{\beta_i}^{l_i}| = 1$ and Lemma 4 (iv).

From (i) it follows immediately that

$$\psi_{\alpha,z,\lambda,g}(i,l_i) = \psi_{\alpha,z,\lambda,f}(i,l_i), \quad l_i \in \mathbb{Z},\ i = 1,\dots,m,$$

which together with (ii) implies

$$G_{\alpha,z}(\lambda,f) \leq G_{\alpha,z}(\lambda,g).$$

In addition,

$$\begin{aligned} G_{\alpha,z}(\lambda, g) &= \sum_{i=1}^{m} \sum_{l \in \mathbb{Z}^m} \lambda_i(l+\beta) \int_{C_{\beta_i}^{l_i}} \varphi_{\alpha_i,z_i}(x_i) g_i(x_i) dx_i \prod_{j \neq i} \int_{C_{\beta_j}^{l_j}} g_j(x_j) dx_j \\ &\leq \sum_{i=1}^{m} \sum_{l \in \mathbb{Z}^m} \lambda_i^* \int_{C_{\beta_i}^{l_i}} \varphi_{\alpha_i,z_i}(x_i) g_i(x_i) dx_i \prod_{j \neq i} \int_{C_{\beta_j}^{l_j}} g_j(x_j) dx_j \\ &= G_{\alpha,z}(\lambda^*, g), \end{aligned}$$

where the inequality is true, since the coefficient of each $\lambda_i(l+\beta)$ is nonnegative because of (iii). □

Next we state an upper bound for the relaxed optimization problem $N(B)$ defined in (29).

Proposition 3 *For every* $B \in \mathbb{R}^m_{++}$,

$$N(B) \leq \sum_{i=1}^{m} \lambda_i^* h(B_i),$$

with N *defined in (29),* λ_i^* *defined in (28), and* h *defined in (26).*

Proof Using Proposition 2 we have that

$$\begin{aligned} N(B) &= \sup_{\alpha \in \mathbb{R}^m} \sup_{z \in \mathbb{R}^m} \sup_{f \in \mathcal{F}^m} \sup_{\lambda \in \Lambda^m} \Big\{ G_{\alpha,z}(\lambda, f) : |\Delta| f_i \leq B_i,\ i = 1, \ldots, m \Big\} \\ &\leq \sup_{\alpha \in \mathbb{R}^m} \sup_{z \in \mathbb{R}^m} \sup_{f \in \mathcal{F}^m} \Big\{ G_{\alpha,z}(\lambda^*, f) : |\Delta| f_i \leq B_i,\ i = 1, \ldots, m \Big\}. \end{aligned}$$

Note that for every $\alpha \in \mathbb{R}^m$, $z \in \mathbb{R}^m$, and $f \in \mathcal{F}^m$ with $|\Delta| f_i = B_i$,

$$\begin{aligned} G_{\alpha,z}(\lambda^*, f) &= \mathbb{E}_\omega \Big[\lambda^*(\omega) \Big\{ \lceil \omega \rceil_z - \lceil \omega \rceil_\alpha \Big\} \Big] \\ &= \sum_{i=1}^{m} \lambda_i^* \mathbb{E}_{\omega_i} \Big[\Big\{ \lceil \omega_i \rceil_{z_i} - \lceil \omega_i \rceil_{\alpha_i} \Big\} \Big] \\ &= \sum_{i=1}^{m} \lambda_i^* D_{\alpha_i,z_i}(f_i) \\ &\leq \sum_{i=1}^{m} \lambda_i^* \mathcal{M}(\alpha_i, f_i), \end{aligned}$$

where D_{α_i,z_i} is defined in Definition 3 and $\mathcal{M}$ in (13). The result now follows from Theorem 1. □

We are now ready to state our main result on the error bound for α-approximations of TU integer recourse functions with independently distributed components of the right-hand side vector ω.

Theorem 2 *Consider the TU integer recourse function Q defined as*

$$Q(z) = \mathbb{E}_\omega \left[\min_y \left\{ qy : Wy \geq \omega - z,\, y \in \mathbb{Z}_+^{n_2} \right\} \right], \quad z \in \mathbb{R}^m,$$

and for every $\alpha \in \mathbb{R}^m$ its α-approximation Q_α defined as

$$Q_\alpha(z) = \mathbb{E}_\omega \left[\min_y \left\{ qy : Wy \geq \lceil \omega \rceil_\alpha - z,\, y \in \mathbb{R}_+^{n_2} \right\} \right], \quad z \in \mathbb{R}^m.$$

Under the assumptions of Section 2, we have for every $\alpha \in \mathbb{R}^m$ and every random right-hand side vector ω with joint density function $f \in \mathcal{F}^m$ that

$$\sup_{z \in \mathbb{R}^m} |Q(z) - Q_\alpha(z)| \leq \sum_{i=1}^m \lambda_i^* h(|\Delta| f_i),$$

where λ_i^* is defined in (28) and h is defined in (26).

Proof Let $\alpha \in \mathbb{R}^m$ and $f \in \mathcal{F}^m$ with $|\Delta| f_i = B_i$, $i = 1, \dots, m$ be given. Then,

$$\sup_{z \in \mathbb{R}^m} |Q(z) - Q_\alpha(z)| =: \mathcal{N}(\alpha, f) \leq \mathcal{N}^*(f) \leq N(B),$$

where the first inequality follows from Lemma 5, and the second from the definition of N in (29). Now the result follows directly from Proposition 3. $\square$

Remark 5 In order to obtain λ_i^* we do not have to compute all possible dual vertices λ^k, $k = 1, \dots, K$. We only have to solve m linear programming problems, since

$$\begin{aligned} \lambda_i^* &= \max_{k=1,\dots,K} \lambda_i^k \\ &= \max_\lambda \{\lambda_i : \lambda W \leq q,\, \lambda \geq 0\} \\ &= \min_y \{qy : Wy \geq e_i,\, y \in \mathbb{R}_+^{n_2}\}, \end{aligned}$$

with e_i denoting the i-th unit vector.

The error bound in Theorem 2 shows that α-approximations are good approximations as long as the total variations of the densities of all random variable in the model are small enough. Moreover, for simple integer recourse models this bound is tight and improves the known bound (3) of [4] with a factor 2.

Corollary 2 *Consider the m-dimensional SIR function*

$$Q(z) = \mathbb{E}_\omega \Big[\min_y \{qy : y \geq \omega - z,\, y \in \mathbb{Z}_+^{n_2}\} \Big], \quad z \in \mathbb{R}^m,$$

and let $B \in \mathbb{R}_{++}^m$ be given. Assume that $q \geq 0$ so that the recourse is sufficiently expensive. Then, for every $\alpha \in \mathbb{R}^m$ there exists $f \in \mathcal{F}^m$ such that $|\Delta| f_i = B_i$, $i = 1, \dots, m$, and

$$\sup_{z \in \mathbb{R}^m} |Q(z) - Q_\alpha(z)| = \sum_{i=1}^m \lambda_i^* h(|\Delta| f_i).$$

Proof For SIR models, the dual feasible region is given by $\{\lambda \in \mathbb{R}_+^{n_2} : \lambda \leq q\}$ so that $\lambda_i^* = q_i \geq 0$. Hence, by Theorem 2, the bound equals

$$\sup_{z \in \mathbb{R}^m} |Q(z) - Q_\alpha(z)| \leq \sum_{i=1}^m \lambda_i^* h(|\Delta| f_i) = \sum_{i=1}^m q_i h(|\Delta| f_i).$$

On the other hand, since for SIR models Q and Q_α are separable, see (2), we have

$$Q(z) - Q_\alpha(z) = \sum_{i=1}^m q_i \mathbb{E}_{f_i} \left[\lceil \omega_i - z_i \rceil^+ - \left(\lceil \omega_i \rceil_{\alpha_i} - z_i \right)^+ \right], \quad z \in \mathbb{R}^m.$$

It appears to be useful to restrict the attention to pdf f_i and real numbers z_i such that f_i vanishes on $(-\infty, z_i]$. Then the '+' operations in the last formula are superfluous, so that (see Remark 4)

$$\begin{aligned} Q(z) - Q_\alpha(z) &= \sum_{i=1}^m q_i \mathbb{E}_{f_i} \Big[\lceil \omega_i - z_i \rceil - \lceil \omega_i \rceil_{\alpha_i} + z_i \Big] \\ &= \sum_{i=1}^m q_i D_{\alpha_i, z_i}(f_i). \end{aligned}$$

Consequently, in order to show that the bound of Theorem 2 is tight, it is sufficient to show that for all $i \in \{1, \ldots, m\}$, $\alpha_i \in \mathbb{R}$ and $B_i \in \mathbb{R}_{++}$ there exist $z_i \in \mathbb{R}$ and $f_i \in \mathcal{F}$ with $f_i(x_i) = 0$ for $x_i \leq z_i$ and $|\Delta| f_i = B_i$ such that

$$D_{\alpha_i, z_i}(f_i) = h(B_i) = \begin{cases} B_i/8, & 0 < B_i \leq 4, \\ 1 - 2/B_i, & B_i \geq 4, \end{cases}$$

and this can be done easily by using the pdf $\hat{f}$ and $\bar{f}$ introduced in (24) and (25). Indeed, if $B_i \in (0, 4]$ then choose $z_i = \alpha_i - 1/2$, so that $\gamma_{\alpha_i, z_i} = 1/2$ and thus $\gamma_{\alpha_i, z_i} \geq 1 - 2/B_i$, and $f_i = \bar{f}$ with $z := z_i$ and $\alpha := \alpha_i$. Then,

$$D_{\alpha_i, z_i}(f_i) = \gamma_{\alpha_i, z_i}(1 - \gamma_{\alpha_i, z_i}) B_i / 2 = B_i / 8.$$

If $B_i \geq 4$ then choose $z_i = \alpha_i - 2/B_i$, so that $\gamma_{\alpha_i, z_i} = 1 - 2/B_i$, and $f_i = \hat{f}$ with $z := z_i$ and $\alpha := \alpha_i$. Then,

$$D_{\alpha_i, z_i}(f_i) = \gamma_{\alpha_i, z_i} = 1 - 2/B_i. \qquad \square$$

6 TU integer recourse models with dependent random right-hand side parameters

In this section we consider the possibility that the random variables in the model are dependent. Again we assume that ω is continuously distributed, but now we assume that the joint density function f is contained in a larger set $\mathcal{H}$, allowing for dependency.

Definition 7 Let $\mathcal{H}$ denote the set of m-dimensional joint density functions f whose conditional density functions $f_i(\cdot|x_{(i)})$ defined as

$$f_i(x_i|x_{(i)}) = f(x)/f_{(i)}(x_{(i)})$$

are contained in $\mathcal{F}$ for all $i = 1, \dots, m$, and $x_{(i)} \in \mathbb{R}^{m-1}$. (As before we use the notation $x_{(i)}$ for the vector x without its i-th component.)

Of course, this definition only makes sense for those i and $x_{(i)}$ for which $f_{(i)}(x_i) > 0$. If $f_{(i)}(x_{(i)}) = 0$, any definition of $f_i(x_i|x_{(i)})$ is good but irrelevant, since in calculating expectations via conditioning its contribution is multiplied by $f_{(i)}(x_{(i)})$, that is by 0.

Using the results from the previous sections we are able to derive an error bound in this case as well.

Theorem 3 *Consider the TU integer recourse function Q defined as*

$$Q(z) = \mathbb{E}_\omega \left[\min_y \left\{ qy : Wy \geq \omega - z,\, y \in \mathbb{Z}_+^{n_2} \right\} \right], \quad z \in \mathbb{R}^m,$$

and for every $\alpha \in \mathbb{R}^m$ its α-approximation Q_α defined as

$$Q_\alpha(z) = \mathbb{E}_\omega \left[\min_y \left\{ qy : Wy \geq \lceil \omega \rceil_\alpha - z,\, y \in \mathbb{R}_+^{n_2} \right\} \right], \quad z \in \mathbb{R}^m.$$

Under the assumptions of Section 2, we have for every $\alpha \in \mathbb{R}^m$ and every random right-hand side vector ω with joint density function $f \in \mathcal{H}$ that

$$\sup_{z \in \mathbb{R}^m} |Q(z) - Q_\alpha(z)| \leq \sum_{i=1}^m \lambda_i^* \mathbb{E}_{\omega_{(i)}} \left[h\left(|\Delta| f_i(\cdot|\omega_{(i)}) \right) \right],$$

where λ_i^* is defined in (28) and h is defined in (26).

Proof We follow the line of proof of the previous section, using the same notation. Obviously, Lemma 5 also holds for $f \in \mathcal{H}$ so that

$$\sup_{z \in \mathbb{R}^m} |Q(z) - Q_\alpha(z)| \leq \sup_{\alpha \in \mathbb{R}^m} \sup_{z \in \mathbb{R}^m} \sup_{\lambda \in \Lambda^m} G_{\alpha,z}(\lambda, f),$$

and similar as in the proof of Proposition 2 we have

$$\begin{aligned} G_{\alpha,z}(\lambda, f) &= \mathbb{E}_f \Big[\lambda(\omega) \Big(\lceil \omega \rceil_z - \lceil \omega \rceil_\alpha \Big) \Big] \\ &= \sum_{i=1}^m \int_{\mathbb{R}^m} \lambda_i(x) \varphi_{\alpha_i, z_i}(x_i) f(x) dx. \end{aligned}$$

However, now we apply conditioning using $f(x) = f_i(x_i|x_{(i)}) f_{(i)}(x_{(i)})$ to obtain

$$\begin{aligned} G_{\alpha,z}(\lambda, f) &= \sum_{i=1}^m \int_{\mathbb{R}^{m-1}} \Big\{ \int_{\mathbb{R}} \lambda_i(x) \varphi_{\alpha_i, z_i}(x_i) f_i(x_i|x_{(i)}) dx_i \Big\} f_{(i)}(x_{(i)}) dx_{(i)} \\ &= \sum_{i=1}^m \int_{\mathbb{R}^{m-1}} G^1_{\alpha_i, z_i} \Big(\hat{\lambda}_i(\cdot|x_{(i)}), f_i(\cdot|x_{(i)}) \Big) f_{(i)}(x_{(i)}) dx_{(i)}, \end{aligned}$$

where $G^1_{\alpha_i,z_i}$ denotes the case $m = 1$ in the definition of $G_{\alpha,z}$ and $\hat{\lambda}_i(\cdot|x_{(i)}) : \mathbb{R} \mapsto \mathbb{R}$ is defined as $\hat{\lambda}_i(x_i|x_{(i)}) = \lambda_i(x)$. Since this function $\hat{\lambda}_i(\cdot|x_{(i)}) \in \Lambda^1$ for all $x_{(i)} \in \mathbb{R}^{m-1}$, we can apply Proposition 3 with $m = 1$, $\alpha = \alpha_i$, $z = z_i$, $\lambda = \hat{\lambda}_i(\cdot|x_{(i)})$ and $f = f_i(\cdot|x_{(i)})$ yielding

$$\begin{aligned} G_{\alpha,z}(\lambda, f) &\leq \sum_{i=1}^m \int_{\mathbb{R}^{m-1}} \lambda_i^* h\left(|\Delta| f_i(\cdot|x_{(i)})\right) f_{(i)}(x_{(i)}) dx_{(i)} \\ &= \sum_{i=1}^m \lambda_i^* \mathbb{E}_{\omega_{(i)}}\Big[h\left(|\Delta| f_i(\cdot|\omega_{(i)})\right)\Big]. \end{aligned}$$ □

Obviously, Theorem 3 generalizes Theorem 2 since $\mathcal{F}^m \subset \mathcal{H}$. If $f \in \mathcal{F}^m$, then the conditional density $f(x_i|x_{(i)}) = f_i(x_i)$ for all $x \in \mathbb{R}^m$, and thus the error bound in Theorem 3 reduces to the one in Theorem 2.

The following example illustrates the impact on the error bound of dependency between the random variables in the model.

Example 1 Let $f \in \mathcal{H}$ be the joint density function of a bivariate normal random vector ω with correlation coefficient ρ. It is well known that $\omega_1|\omega_2 = x_2$ follows a normal distribution with variance $(1-\rho^2)\sigma_1^2$. Hence, for $i = 1, 2$, and $x_{(i)} \in \mathbb{R}$,

$$|\Delta| f_i(\cdot|x_{(i)}) = \frac{2}{\sqrt{2\pi(1-\rho^2)\sigma_i^2}} = \sqrt{\frac{2}{\pi(1-\rho^2)\sigma_i^2}}.$$

This implies that the error bound in Theorem 3 for this particular joint density function equals

$$\sum_{i=1}^2 \lambda_i^* \mathbb{E}_{\omega_{(i)}}\left[h\left(|\Delta| f_i(\cdot|\omega_{(i)})\right)\right] = \sum_{i=1}^2 \lambda_i^* h\left(\sqrt{\frac{2}{\pi(1-\rho^2)\sigma_i^2}}\right).$$

For small values of ρ and sufficiently large values of σ_1^2 and σ_2^2 this error bound is equal to

$$\sum_{i=1}^2 \frac{\lambda_i^*}{8}\sqrt{\frac{2}{\pi(1-\rho^2)\sigma_i^2}} = \frac{1}{\sqrt{1-\rho^2}} \sum_{i=1}^2 \frac{\lambda_i^*}{8}\sqrt{\frac{2}{\pi\sigma_i^2}}.$$

If $\rho = 0$ the bound reduces to the error bound of Theorem 2 (the case where ω_1 and ω_2 are independent). If $|\rho| \leq 0.4$, we have that $1/\sqrt{1-\rho^2} \leq 1.1$. Hence, only for high correlation values $|\rho|$ the error bound substantially increases compared to the case where ω_1 and ω_2 are independent.

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